Putting the unpredictable into equations

Concept of statistical physics.

The number of nuclei present in the samples typically handled is so large (> 10^{15}) that it requires a change of approach. This new physics, called statistical physics, makes use of probabilities and the laws of large numbers.

Laws of probability.

1. Number of observed decays.

Let ΔN be the number of decays observed during duration Δt.

Note: Δ always represents the difference betwwen a final state and an initial state.

a. Influence of the number of nuclei present.

The more radioactive nuclei in the sample, the more decays are observed.

b. Influence de la durée d'observation.

The longer the duration of the observation, the more decays are observed.

c. Influence de la nature du noyau.

Not all nuclei have the same propensity to decay. The ability of a nucleus to decay more or less rapidly is expressed by its decay constant, denoted by λ.

Note: The unit of the decay constant is homogeneous to the inverse of time.

The higher the decay constant, the more decays are observed.

d. Recap.

$$\Delta N = -\lambda N \Delta t$$

Note: λ, N and Δt are positive, while ΔN is negative (the number of nuclei present is decreasing), thus the presence of the "-" sign.
This formula is valid only for an important number of radioactive nuclei (> 10^5).

2. Law of probability.

a. A bit of mathematics – the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dt}$$

b. Experimental sciences and differential calculus.

Multiple factors can be involved when analyzing a physical phenomenon. It is therefore essential to state explicitely which are varied.

derivative with respect to x: $\frac{df}{dx}$, or f'(x) second derivative: $\frac{d^2f}{dx^2}$, or f"(x)

derivative with respect to t: $\frac{df}{dt}$, or $\dot{f}(t)$ second derivative: $\frac{d^2f}{dt^2}$, or $\ddot{f}(t)$

c. Differential equation related to the decay of a radioactive sample.

$\frac{\Delta N}{\Delta t} = -\lambda N.$

When reducing duration Δt to the smallest possible, $\lim_{\Delta t \to 0} \frac{\Delta N}{\Delta t} = \frac{dN}{\mathrm{d}t}$

Thus the differential equation: $$\frac{dN}{\mathrm{d}t} = -\lambda N \qquad (1)$$

Radioactive decay.

1. A bit more maths.

a. The exponential function.

$$f(x) = e^x$$
$$(e^u)' = u'e^u$$
$$e^0 = 1$$

b. The natural logarithme function.

$$f(x) = lnx$$
$$(lnu)' = \frac{u'}{u}$$
$$lnab = lna + lnb$$
$$ln\frac{a}{b} = lna - lnb$$
$$ln(a^b) = blna$$
$$ln1 = 0$$

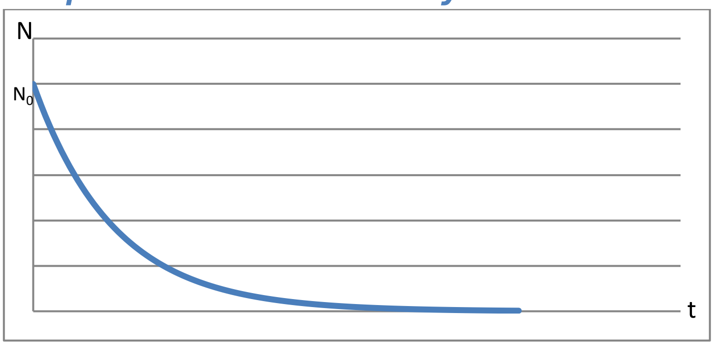
c. e vs. ln.

$$e^{lnx} = x$$
$$ln(e^x) = x$$

2. « Solving » the differential equation ».

The solution of the differential equation (1) is $N(t) = N_0e^{-\lambda t}$
This is the law of radioactive decay.

3. Graph of radioactive decay.



Note: The curve has the same shape whatever the initial number of radioactive nuclei, N_0.

4. Half-life of a radioactive sample.

a. Definition.

The half-life of a radioactive sample is the duration after which half of the nuclei of the sample have decayed.

b. Theoretical determination.

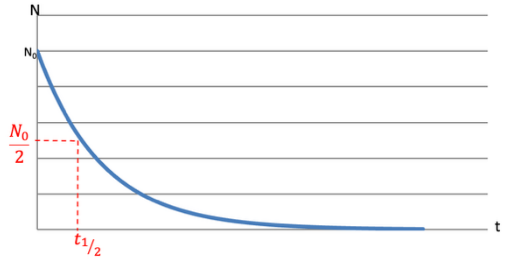
$$N\left(t_{1/2}\right) = N_0e^{-\lambda t_{1/2}} = \frac{N_0}{2} \Rightarrow \frac{1}{2} = e^{-\lambda t_{1/2}} \Longrightarrow 2 = e^{\lambda t_{1/2}} \Longrightarrow ln2 = \lambda t_{1/2}$$

$$\Longrightarrow t_{1/2} = \frac{ln2}{\lambda}$$

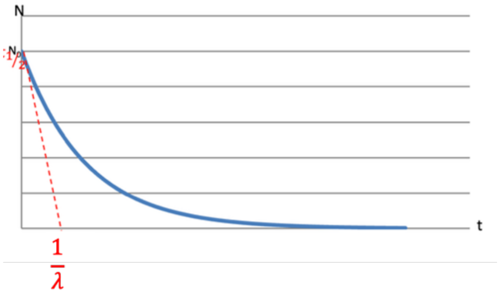
Note: $N\left(nt_{1/2}\right) = \frac{N_0}{2^n}$
It can be considered that a sample has fully decayed after a duration of $5t_{1/2}$.

c. *Graphical determination.*

- By projection:



- By the tangent at origin:



Dating a radioactive sample.

1. *Concept of radioactive activity.*

There is no such thing as a radioactive nucleus counter. However, it is possible to count the number of particles emitted by a radioactive sample during a time interval Δt, i.e., the number of decays occurring during that time interval.

The rate at which these decays occur is the activity A of the sample.

$$A(t) = -\frac{dN(t)}{dt}$$

Notes: $\frac{dN(t)}{dt}$ is the derivative of N(t) in respect to time. It is negative ⇒ The "-" signe gives a positive value for activity.

The activity of a radioactive sample is measured in bequerels (Bq), with a Geiger counter, one bequerel corresponding to one decay per second.

Background radiation contributes additional counts to the detector, increasing the measured count rate. To obtain the true count rate from the source, the background count rate is measured separately and subtracted from the total count rate.

This correction is especially important when the source activity is low.

2. *Evolution of the activity of a sample over time.*

$$N(t) = N_0 e^{-\lambda t} \Longrightarrow A(t) = -\frac{dN(t)}{dt} = \lambda N_0 e^{-\lambda t} = A_0 e^{-\lambda t}$$

Note: $A(t) = \lambda N(t)$. The activity of a sample is proportional to the number of radioactive nuclei of the sample ⇒ Studying A(t) is therefore equivalent to studying N(t).

3. *Application to dating.*

Measuring the activity of a radioactive sample allows for an estimation of its age.

$$A(t) = A_0 e^{-\lambda t} \Longrightarrow \frac{A(t)}{A_0} = e^{-\lambda t} \Longrightarrow ln\frac{A(t)}{A_0} = -\lambda t$$

$$\Longrightarrow t = \frac{1}{\lambda} ln\frac{A_0}{A(t)}$$