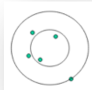
From measurements to a result

Measuring

No experimental measurement technique is rigorously reliable. Every measurement is subject to error, so there is no such thing as an exact measurement. The ‘true’ value of a measurement is technically inaccessible. An experimental result must therefore always take this error into account.

1. Sources of error

a. Random error



An error is said to be random when a large number of measurements of the same quantity under the same conditions show that the different results are distributed on either side of the mean value obtained.
The origin of a random error is generally linked to the inaccuracy of the measuring device (discrete graduations) and to the ‘lack of rigour’ of the experimenter (this contribution is, however, difficult to quantify, but can sometimes be estimated).

Note: Sometimes the random error can be linked to the phenomenon being measured itself, if it is particularly unstable. Weather forecasts, for example, are based on data that is highly unstable and subject to considerable variability.

When an experiment is subject to random error, the results of repeated measurements under the same conditions are far apart. However, this does not prevent the mean value from being correct.

b. Systematic error

An error is said to be systematic when it always takes the same value on each repeated measurement. It always affects the result in the same direction.
The origin of a systematic error is generally a defect in the measuring equipment or the experimental method. It is difficult to see in the raw data, but can sometimes be easily corrected when the data is processed.

When an experiment is marred by a systematic error, the results of repeated measurements under the same conditions are very close. The instrument used is therefore accurate. However, the measured value is incorrect.

2. Quality vs. accuracy

Let X be the quantity measured

a. Standard uncertainty

The standard uncertainty, or uncertainty of measurement, is a parameter associated with the result of the measurement, enabling its quality to be judged.
Noted U(X), the standard uncertainty is a number of the same unit as the measurement made. It is always written with a single significant figure (rounded upwards).
The experimental result will be written as: $\boldsymbol{X = X_{measured} + U(X)}$

b. Some common uncertainties

Generally speaking, the standard uncertainty of an instrument corresponds to half its gradation.
When using chemistry equipment, standard uncertainties are generally supplied by the manufacturer and indicated on the equipment.

Ex: Measure a length l1 = 15.2 cm using a ruler graduated in mm.
$U(l_1)$ = 0.5 mm, and the result of the measurement is as follows: $l = 152 \pm 0{,}5\ mm$
If the ruler used is graduated in half-mm, we have $U(l_2)$ = 0,25 mm, and $l = 152 \pm 0{,}3\ mm$.
We measure a length l_3 = 384000 km with a 1 m long stick: $l = 384000000 \pm 0{,}5\ m$

c. Relative uncertainty

To determine the accuracy of a measurement, the standard uncertainty is no longer sufficient. We need to calculate its relative uncertainty, or precision, $\boldsymbol{\frac{\Delta X}{X} = \frac{U(X)}{X}}$. This is a number without a unit, often presented as a percentage.

Ex: $\left(\frac{\Delta l}{l}\right)_1 = \frac{0{,}5}{152} = 3{,}3.10^{-3}$

$\left(\frac{\Delta l}{l}\right)_2 = \frac{0{,}25}{152} = 1{,}6.10^{-3}$

The second measurement is more accurate than the first.

$\left(\frac{\Delta l}{l}\right)_3 = \frac{0{,}5}{384000000} = 1{,}3.10^{-9}$ *The quality of the third measurement has a lower quality than the other two, but is more accurate.*

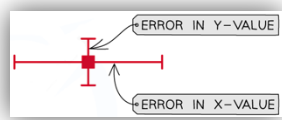
Processing

Because of the existence of systematic and random errors, using raw data can lead to false conclusions. The **BEST WAY** to highlight these errors is visually, through the **TRACE OF A GRAPH**.

1. Representation of experimental points

The pairs of measurement points are placed on a graph with a carefully chosen coordinate system. The points should be represented by a cross.
This cross is then surrounded by an uncertainty rectangle, obtained from the absolute uncertainty of each of the measured quantities. Because of the measurement uncertainties, the pairs obtained are unlikely to be 'true' pairs. However, by evaluating the absolute uncertainties, it can be stated that the 'true' pair lies within the uncertainty rectangle.

Note: The axes MUST be named (size represented + unit).
The axes MUST be graduated.
The graph MUST have a title.
Points should NEVER be joined by line segments.



2. Treatment of outliers

By viewing the resulting cloud of points, a trend can be identified, generally associated with a known mathematical function. Any point that does not follow this trend needs to be re-examined, either through a new measurement or by eliminating it from the series.

Note: Many scientific discoveries have been made as a result of these aberrant points. Their existence calls into question a theoretical model, and leads to a new model that includes this point... and the previous results.
For example, Wien's law worked very well at long wavelengths, but gave false predictions for UV radiation. Max Planck's quantification of light made it possible to take this particularity into account, while still giving accurate results at long wavelengths.

3. Linearisation

When the trend shown on the graph is not linear, the expression of the quantities represented on the axes can be modified to produce a straight line. This operation, known as linearisation, makes it easier to use the graph.
Once the straight line has been obtained, a common source of systematic error can often be identified. Indeed, many phenomena can be modelled by a linear relationship, i.e. by a straight line passing through the origin. The fact that the line does not pass through the origin is therefore a sign of systematic error, which will be eliminated when the graph is used.

4. Adding a trend line

Once the curve has been linearised, a trend line is drawn, in the form of an average straight line.
Any line passing through all the uncertainty rectangles is considered to be the mean line. If this is not possible, the measurement protocol must be repeated, either by carrying out new measurements with greater rigour, or by reassessing the standard uncertainties.

Note: Because of measurement uncertainties, several average lines can sometimes be drawn. Each of them is valid.

Modelling

1. From slope to value

When studying a phenomenon, the magnitude to be determined is often linked to the slope of the mean line. Calculating this slope gives access to it.

Note: *The slope of a line is independent of its y-intercept. A systematic error therefore has no effect on its value, and is therefore eliminated when determining it.*

2. Uncertainty of the calculated value

a. A few rules for calculating uncertainties

- If $X = a + b$, or $X = a - b$, standard uncertainty varies:

$$\boldsymbol{U(X) = \sqrt{\left(U(a)\right)^2 + \left(U(b)\right)^2}}$$

Ex: $a = 204{,}3 \pm 0{,}3\ mm$ and $b = 191{,}2 \pm 0{,}3\ mm \Rightarrow \begin{cases} G_+ = 395{,}5 \pm 0{,}4\ mm \\ G_- = 13{,}1 \pm 0{,}4\ mm \end{cases}$

- If $X = a^n \times b^m$, or $X = \frac{a^n}{b^m}$, relative uncertainty varies:

$$\boldsymbol{\frac{\Delta X}{X} = \sqrt{n\left(\frac{\Delta a}{a}\right)^2 + m\left(\frac{\Delta b}{b}\right)^2} \Rightarrow U(X) = \frac{\Delta X}{X} \times X}$$

Ex: $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}} \Rightarrow \frac{\Delta T}{T} = \sqrt{\frac{1}{2}\left(\frac{\Delta m}{m}\right)^2 + \frac{1}{2}\left(\frac{\Delta k}{k}\right)^2}$

b. Presentation of the final value

After determining the standard uncertainty of the quantity under study, the result is presented in the form of an interval:

$$\boldsymbol{X = X_{calculated} \pm U(X)}$$
$$\boldsymbol{X \in [X_{calculated} - U(X); X_{calculated} + U(X)]}$$

3. Comparison with a theoretical value

Note: *The same notation is used to designate an uncertainty or a deviation. What is calculated depends on the context.*

a. Difference between measured value and theoretical value

When a value measured experimentally is compared with a reference value, a deviation is determined.

The absolute deviation is calculated as follows: $\boldsymbol{\Delta X = \left|X_{ref} - X_{exp}\right|}$

The relative deviation is calculated as follows: $\boldsymbol{\frac{\Delta X}{X} = \left|\frac{X_{ref} - X_{exp}}{X_{ref}}\right|}$

b. z-score

The z-score is the result of comparing the absolute deviation with the standard uncertainty:

$$z = \boldsymbol{\frac{\Delta X}{U(X)} = \frac{\left|X_{ref} - X_{exp}\right|}{U(X)}}$$

It represents an assessment of the agreement between the measurement result and the reference value of the quantity G.

Note: The experimental result is considered to be compatible with the reference value when the z-score is less than 2.

Improving results through collaboration

When determining the value of a quantity, certain systematic errors may go undetected. To reduce this error, the result must be compared with that of other independent experiments. A statistical study of this new series will then enable us to approach a theoretical value.

1. Some basics in statistics

a. Mean value

The mean $\bar{X}$ of the n measurements in a sample is the best estimator of the sample of n independent measurements: $\boldsymbol{\bar{X} = \frac{X_1+X_2+X_3+\cdots+X_n}{n} = \frac{\sum_1^n G_i}{n}}$

b. Standard deviation

The best estimate of the dispersion of a series of measurements is measured by the standard deviation σ:

$$\boldsymbol{\sigma_{n-1} = \sqrt{\frac{\sum_1^n (X_i - \bar{X})}{n-1}}}$$

Note: The mean and standard deviation of a series of measurements can be obtained using the calculator's STAT mode..

2. Confidence interval

A confidence interval is an interval in which the value sought has a certain probability of being found.

Ex: The confidence interval at 95% is the interval in which the probability to find the sought value is 95 %.

The size of the confidence interval is determined from the expanded uncertainty $U(X)_{\%}$:

$$\boldsymbol{U(X)_{\%} = t_{\%}\frac{\sigma_{n-1}}{\sqrt{n}}}$$

$t_{\%}$ is a coefficient called the Student coefficient. Its value depends on the number of measurements in the series and the desired confidence level:

n	2	3	4	5	6	7	8	9	10
$t_{95\%}$	12,7	4,3	3,18	2,78	2,57	2,45	2,37	2,31	2,26
$t_{99\%}$	63.7	9,93	5,84	4,6	4,03	3,71	3,5	3,36	3.25

n	12	14	16	18	20	30	50	100	∞
$t_{95\%}$	2,2	2,16	2,13	2,11	2,09	2,04	2,01	1,98	1.96
$t_{99\%}$	3,11	3,01	2.95	2,9	2,86	2,76	2,68	2,63	2.57

Note: Usually, we consider $t_{95\%}$ = 2 et $t_{99\%}$ = 3

3. Presentation of the result

The definite result of the experiment is presented as: $\boldsymbol{X = \bar{X} \pm U(X)_{\%}}$